Homework 5: Due Friday, February 21

Problem 1: Let $r \in \mathbb{R}$ with $r \neq 1$. Use induction to show that

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$.

Problem 2: Let f_n be the sequence of Fibonacci numbers, i.e. $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Show that $\gcd(f_{n+1}, f_n) = 1$ for all $n \in \mathbb{N}$.

Problem 3: Let $a, b \in \mathbb{N}^+$ and let $d = \gcd(a, b)$. Since d is a common divisor of a and b, we may fix $k, \ell \in \mathbb{N}$ with $a = kd$ and $b = \ell d$. Let $m = k\ell d$.
a. Show that $a \mid m$, $b \mid m$, and $dm = ab$.
b. Suppose that $n \in \mathbb{Z}$ is such that $a \mid n$ and $b \mid n$. Show that $m \mid n$.
Note: The number m is called the *least common multiple* of a and b and is written as $\operatorname{lcm}(a, b)$. Since $dm = ab$ from part (a), it follows that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$. Using this together with the Euclidean Algorithm, we can quickly compute least common multiples.

Problem 4: Define a function $\sigma \colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $\sigma(n)$ be the sum of all positive divisors of n. In other words, if $Div(n) \cap \mathbb{N}^+ = \{d_1, d_2, \dots, d_k\}$, then

$$\sigma(n) = \sum_{i=1}^{k} d_i.$$

For example, $\sigma(6) = 1 + 2 + 3 + 6 = 12$.
a. Give a closed form formula for $\sigma(p^n)$ whenever $p \in \mathbb{N}^+$ is prime and $n \in \mathbb{N}^+$.
b. Show that $\sigma(ab) = \sigma(a) \cdot \sigma(b)$ whenever $a, b \in \mathbb{N}^+$ satisfy $\gcd(a, b) = 1$.
c. Use parts (a) and (b) to give a formula for $\sigma(n)$ in terms of the prime factorization of n.

Problem 5:
a. Prove that if $d, n \in \mathbb{N}^+$ and $d \mid n$, then $2^d - 1 \mid 2^n - 1$.
b. Prove that if $n \in \mathbb{N}^+$ and $2^n - 1$ is prime, then n is prime.
Note: Primes of the form $2^p - 1$, where p is prime, are called *Mersenne primes*. For example, $3 = 2^2 - 1$ is a Mersenne Prime, as is $7 = 2^3 - 1$. Notice that $2^4 - 1 = 15$ is not prime (which follows from the contrapositive of part (b), since 4 is not prime). It turns out that although 11 is prime, the number $2^{11} - 1$ is not prime, so the converse of part (b) is false. It is an open question whether there are infinitely many Mersenne primes.

Problem 6: A number $n \in \mathbb{N}^+$ is called perfect if $\sigma(n) = 2n$. Since we always have $n \mid n$, notice that this is the same as saying that the sum of the *proper* divisors of n equals n. For example, 6 is perfect because $\sigma(6) = 12 = 2 \cdot 6$, and notice that $6 = 1 + 2 + 3$. Show that if $2^p - 1$ is a Mersenne prime, then $2^{p-1}(2^p - 1)$ is perfect.
Cultural Aside: Euler proved a partial converse by showing that every *even* perfect number must equal $2^{p-1}(2^p - 1)$ for some Mersenne prime $2^p - 1$. The existence of odd perfect numbers is an open question.