

## Homework 16: Due Friday, May 8

**Problem 1:** In this problem, you will finish the proof of Proposition 6.6.3. Let  $m = pq$  where  $p, q \in \mathbb{N}^+$  are distinct primes. Let  $k, \ell \in \mathbb{N}^+$  with  $k\ell \equiv 1 \pmod{\varphi(m)}$ . Show that  $(a^k)^\ell \equiv a \pmod{m}$  for all  $a \in \mathbb{Z}$ .

*Hint:* It suffices to show that both  $(a^k)^\ell \equiv a \pmod{p}$  and  $(a^k)^\ell \equiv a \pmod{q}$ .

**Problem 2:** Show that

$$\sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

for all  $n \in \mathbb{N}^+$  by interpreting the left-hand side as a certain Riemann sum.

*Note:* You can find a direct inductive proof of this fact in the proof of Proposition 2.1.8.

**Problem 3:** Let  $n, k \in \mathbb{N}$  with  $0 \leq k < n$ .

a. Show that if  $0 \leq k < \frac{n-1}{2}$ , then  $\binom{n}{k} < \binom{n}{k+1}$ .

b. Show that if  $\frac{n-1}{2} < k < n$ , then  $\binom{n}{k} > \binom{n}{k+1}$ .

c. Show that if  $k = \frac{n-1}{2}$ , then  $\binom{n}{k} = \binom{n}{k+1}$ .

*Note:* This problem says that any row of Pascal's triangle is strictly increasing until the middle term(s), and then strictly decreasing afterwards. I suggest that you start by looking at the quotient  $\binom{n}{k+1}/\binom{n}{k}$ .

**Problem 4:** Show that there are arbitrarily large gaps in the primes, i.e. that for every  $n \in \mathbb{N}^+$ , there exist  $n$  consecutive composite numbers.

*Hint:* Factorials are your friends.

*Interlude:* A natural number  $n \in \mathbb{N}^+$  is called *square-free* if it is not divisible by the square of any natural number greater than or equal to 2. It is straightforward to show that  $n$  is square-free if and only if it is not divisible by  $p^2$  for every prime  $p$ . Using the Fundamental Theorem of Arithmetic, it follows that  $n$  is square-free if and only if either  $n = 1$ , or  $n$  is the product of distinct primes.

**Problem 5:** Given  $n \in \mathbb{N}^+$ , show that  $|\{k \in [n] : k \text{ is square-free}\}| \leq 2^{\pi(n)}$ .

**Problem 6:** It is possible to show that for all  $n \in \mathbb{N}^+$ , there exist unique  $k, m \in \mathbb{N}^+$  such that  $n = k^2m$  and such that  $m$  is square-free (essentially you factor  $n$  into primes, and form  $m$  by pulling out one of each prime that occurs an odd number of times). Use this result and the previous problem to show that

$$\pi(n) \geq \frac{\ln n}{2 \ln 2}$$

for all  $n \in \mathbb{N}^+$ .