Homework 9 : Due Monday, March 10

Problem 1: Let $m, n \in \mathbb{N}^+$ with $m \leq n$. If σ is a permutation of $[n]$, then we say that $i \in [n]$ is a fixed point of σ if $\sigma(i) = i$. How many permutations of $[n]$ have exactly m fixed points?

Problem 2: On Homework 7, you showed that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$. Now show that

$$c(n, n-2) = 2 \cdot \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.

Problem 3: Let $n \in \mathbb{N}^+$.
a. How many ways are there to break up $3n$ (distinct) people into n groups of size 3, where there is no ordering amongst the groups (so all that matters is the people who are grouped together)?
b. How many permutations of $[3n]$ consist of n disjoint 3-cycles?
Note: Simplify your answers as much as possible.

Interlude: Given $n \in \mathbb{N}^+$, define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. The rational numbers H_n are called *harmonic numbers*. It can be shown that as n gets large, then H_n gets arbitrary large and in fact H_n is reasonably close to $\ln n$ (roughly, H_n can viewed as a certain Riemann sum approximating $\int_1^n \frac{1}{x}\, dx$, which is where the $\ln n$ comes from). More precisely, it can be shown that

$$\lim_{n \to \infty} (H_n - \ln n)$$

exists and equals a number $\gamma \approx .5772156649\ldots$ called the Euler-Mascheroni constant (remarkably, it is still not known whether γ is irrational). Thus, when n is large, an extremely good approximation to H_n is

$$H_n \approx \ln n + \gamma$$

Problem 4: Let $n \in \mathbb{N}^+$.
a. Suppose that $k \in \mathbb{N}^+$ with $n + 1 \leq k \leq 2n$. Show that the number of permutations of $[2n]$ containing a k-cycle is $\frac{(2n)!}{k}$.
b. Use part a and the above discussion to show that when n is large, the fraction of permutations of $[2n]$ containing a cycle of length at least $n + 1$ is approximately $\ln 2$.

Problem 5: Suppose that σ is a permutation of $[n]$. Define an $n \times n$ matrix $M(\sigma)$ by letting

$$M(\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

a. Let $n = 4$, let $\sigma = (1\ 2\ 3)(4)$ and let $\tau = (1\ 2)(3\ 4)$. Write down $M(\sigma)$, $M(\tau)$, and $M(\sigma \circ \tau)$.
b. Show that $M(\sigma \circ \tau) = M(\sigma) \cdot M(\tau)$ for all permutations σ and τ of $[n]$ (not just those in part a).