Homework 3 : Due Wednesday, February 5

Problem 1: Show that if $4 \mid a$, then a is the difference of two perfect squares, i.e. there exist $b, c \in \mathbb{Z}$ with $a = b^2 - c^2$.

Problem 2: Define a sequence recursively by letting $a_0 = 42$ and letting

$$a_{n+1} = a_n^2 - 3a_n + 14$$

for all $n \in \mathbb{N}$. Show that $7 \mid a_n$ for all $n \in \mathbb{N}$.

Problem 3: Find a formula for

$$\sum_{k=1}^{n} (-1)^{k-1}(2k-1) = 1 - 3 + 5 - 7 + 9 - \cdots + (-1)^{n-1}(2n-1)$$

and prove that your formula is correct for all $n \in \mathbb{N}^+$.

Problem 4: Define a sequence by letting $a_0 = -1$, $a_1 = 7$, and $a_n = 2a_{n-1} + 4a_{n-2}$ for all $n \geq 2$. Show that $a_n \geq 3^n$ for all $n \in \mathbb{N}^+$.

Note: For the next two problems, let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

Problem 5: Show that

$$f_{n+1}f_{n-1} = f_n^2 + (-1)^n$$

for all $n \in \mathbb{N}^+$.

Problem 6: A friend tries to convince you that $2 \mid f_n$ for all $n \geq 3$. Here is their argument using strong induction. For the base case, notice that $f_3 = 2$, so $2 \mid f_3$. For the inductive step, suppose that the $n \geq 4$ and we know the result for all k with $3 \leq k < n$. Since $f_n = f_{n-1} + f_{n-2}$ and we know by induction that $2 \mid f_{n-1}$ and $2 \mid f_{n-2}$, it follows that $2 \mid f_n$. Therefore, $2 \mid f_n$ for all $n \geq 3$.

Now you know that your friend's argument must be wrong because $f_7 = 13$ and $2 \nmid 13$. Pinpoint the fundamental error. Be as explicit and descriptive as you can.