Homework 15: Due Friday, May 2

Problem 1: Let G be a graph. A *vertex cover* in G is a set $S \subseteq V$ such that every edge in G has at least one endpoint in S. Show that if M is a matching in G and S is a vertex cover in G, then $|M| \leq |S|$.

Problem 2: Let G be a graph. Let M be a maximum matching in G and suppose that M has k edges. Show that if L is a maximal matching in G, then L has at least $\frac{k}{2}$ many edges.

Problem 3: Given $n \in \mathbb{N}$ with $n \geq 3$, recall that C_n is the graph with vertex set $[n]$ where 1 and n are adjacent and also k and $k+1$ are adjacent whenever $1 \leq k \leq n-1$ (so C_n is just a cycle of length n).
a. Determine the size of a maximum matching in C_{6m} for each $m \in \mathbb{N}^+$.
b. Determine the size of a smallest possible maximal matching in C_{6m} for each $m \in \mathbb{N}^+$.

Problem 4: Show that a tree has at most one perfect matching.
Hint: Given two perfect matchings, think about the symmetric difference.

Problem 5: Fix a finite graph G. Consider the following game with two players. Player 1 starts by picking a vertex v_1 of G. Player 2 responds by picking a new vertex v_2 adjacent to v_1. Player 1 then picks a vertex $v_3 \notin \{v_1, v_2\}$ which is adjacent to v_2. This process continues so that at each stage, the corresponding player picks $v_{k+1} \notin \{v_1, v_2, \ldots, v_k\}$ which is adjacent to v_k. The game ends when a player is unable to make a valid move, in which case the other player is declared the winner.
a. Suppose that G has a perfect matching. Explicitly describe a winning strategy for Player 2 and explain why it works.
b. Suppose that G does not have a perfect matching. Explicitly describe a winning strategy for Player 1 and explain why it works.