Homework 14 : Due Wednesday, April 23

Problem 1: Recall the graph Q_n from Problem 4 on Homework 10. Show that Q_n is bipartite for each $n \in \mathbb{N}^+$.

Problem 2: Let G be the following graph. Let $V_G = \mathcal{P}_2([5])$, i.e. the vertices of G are the subsets of $[5]$ of cardinality 2. For the edge set, let

$$E = \{\{u, w\} : u \text{ and } w \text{ are disjoint sets}\}$$

i.e. we include an edge with endpoints u and w exactly when u and w are 2-element subsets of $[5]$ that do not share a common element. Determine, with justification, the value of $\chi(G)$.

Problem 3: Let T be a finite tree with $n \geq 2$ vertices.
a. Show that T is bipartite.
b. Show that given any proper 2-coloring of the vertices of T, there exists a leaf in the larger of the two color classes (and a leaf in each if both colors are used the same number of times).
Hint for b: Think about the sum of the degrees in each color class.

Problem 4: Let $n \in \mathbb{N}$ with $n \geq 2$. Notice that the number of graphs with vertex set $[n]$ is

$$2^{\binom{n}{2}}$$

(where the $\binom{n}{2}$ is in the exponent) because for each of the $\binom{n}{2}$ many pairs of vertices, we need to decide whether there is an edge with those endpoints. Show that the number of graphs with vertex set $[n]$ where every vertex has even degree is

$$2^{\binom{n-1}{2}}$$

(where the $\binom{n-1}{2}$ is in the exponent).

Problem 5: Let G be a connected graph, and let P and Q be two paths of maximum length in G (so both P and Q have common length k, and there is no path of length greater than k). Show that P and Q have a common vertex.

Problem 6: Let $k \in \mathbb{N}^+$. A k-critical graph is a graph G such that $\chi(G) = k$ but $\chi(H) < k$ for any proper subgraph H of G, i.e. G has chromatic number k, but whenever we delete at least one vertex/edge, the chromatic number becomes strictly smaller. For example, every odd cycle is 3-critical (and in fact these are the only 3-critical graphs).
a. Show that K_n is an n-critical graph for each $n \in \mathbb{N}^+$.
b. Suppose that G is a k-critical graph. Show that $d(v) \geq k - 1$ for all $v \in V$.
Hint for b: Suppose that $v \in V$ is such that $d(v) \leq k - 2$. Think about the graph $G - v$.