Homework 12 : Due Monday, April 14

Problem 1: Given a graph G, we defined $\overline{G}$ in Problem 5 on Homework 11.
a. Let $n \geq 2$. Let G be a graph on n vertices with at least n edges. Show that G contains a cycle.
b. Give an example of graph on 4 vertices such that neither G nor $\overline{G}$ contains a cycle.
c. Show that if G is a graph on $n \geq 5$ vertices, then at least one of G or $\overline{G}$ contains a cycle.

Problem 2: Let T be a finite tree with n vertices. Let a_T be the average degree of the vertices (i.e. the result of summing the degrees of the vertices and dividing by n).
a. Show that $a_T < 2$.
b. Show that if T has a vertex of degree ℓ, then T has at least ℓ leaves.

Problem 3: For each of the following, either prove or find a counterexample.
a. Deleting a vertex of maximum degree in a finite graph G cannot increase the average degree.
b. Deleting a vertex of minimum degree in a finite graph G cannot decrease the average degree.

Problem 4: A saturated hydrocarbon is a molecule C_kH_ℓ in which every carbon atom C has four bonds, every hydrogen atom H has one bond, and no sequence of bonds forms a cycle. Show that $\ell = 2k + 2$ in any saturated hydrocarbon.
Hint: Form a graph and determine the sum of the degrees.

Problem 5: Let T be a finite tree with at least two vertices and such that $d(v) \geq 3$ whenever v is adjacent to a leaf. Show that there exist two leaves u and w of T that share a common neighbor.
Hint: Start by considering a longest possible path in G.

Problem 6: Let G be a finite graph with the property that $d(v) \geq 3$ for all $v \in V$. Prove that G has a cycle of even length.
Hint: See the hint for problem 5.