Homework 10 : Due Friday, April 4

Problem 1: Let $n \in \mathbb{N}^+$.
a. Evaluate
$$\sum_{k=0}^{n} 3^k \cdot c(n,k)$$
b. Evaluate
$$\sum_{k=0}^{n} 3^k \cdot s(n,k)$$
Note: Simplify your answers as much as possible.

Problem 2: Show that if A and B are countable sets, then $A \times B$ is countable.

Problem 3:
a. Recall that $\{0,1\}^*$ is the set of all finite sequences of 0's and 1's (of any finite length). Show that $\{0,1\}^*$ is countable.
b. Let S be the set of all infinite sequences of 0's and 1's (so an element of S looks like $11000101110\dots$). Show that S is uncountable.

Problem 4: Fix $n \in \mathbb{N}^+$. Consider the graph Q_n defined as follows. Let the vertex set V be the set of all sequences of 0's and 1's of length n (so for example, if $n = 3$, then one vertex is 010 and another is 111). Let E be the set of all pairs $\{u, v\}$ such that u and v differ in exactly one coordinate (so for example when $n = 3$ there is edge with endpoints 001 and 101). The graph Q_n is called is called the n-cube.
a. Draw the graphs Q_1, Q_2, and Q_3.
b. Write down the adjacency matrix for Q_3 (clearly indicate the ordering of the vertices that you are using).
c. Let $U = \{0000, 0100, 1110, 1001, 1111\}$. Draw $Q_4[U]$.
d. Determine $d(u)$ for each $u \in Q_n$.
e. Determine the number of vertices and edges in Q_n.

Problem 5: Let G be a finite graph. Explain why the number of 1's in any adjacency matrix of G equals the number of 1's in any incidence matrix of G.

Problem 6: Let G be a finite graph with $|V| \geq 2$. Show that there exists $u, w \in V$ with $u \neq w$ such that $d(u) = d(w)$.