Homework 9 : Due Monday, March 12

Problem 1: Let $n \in \mathbb{N}^+$.
a. Evaluate

$$\sum_{k=0}^{n} c(n,k)$$

b. Evaluate

$$\sum_{k=0}^{n} 2^k c(n,k)$$

Problem 2: On Homework 7, you showed that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$. Now show that

$$c(n, n-2) = 2 \cdot \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.

Problem 3: Let $\ell, n \in \mathbb{N}^+$ with $\ell \leq n$. If σ is a permutation of $[n]$, then we say that $i \in [n]$ is a fixed point of σ if $\sigma(i) = i$. How many permutations of $[n]$ have exactly ℓ fixed points?

Problem 4: How many different ways can you place seven distinct ornaments on three identical circular wreaths? Allow the possibility that some wreaths have no ornaments on them.

Problem 5: Let $n \in \mathbb{N}^+$.
a. How many ways are there to break up $3n$ people into n groups of size 3 (where there is no ordering amongst the groups)? Simplify your answer as much as possible.
b. How many permutations of $[3n]$ consist of n distinct 3-cycles?
c. Explain why your answers in parts a and b are different.

Problem 6: Suppose that σ is a permutation of $[n]$. Define an $n \times n$ matrix $M(\sigma)$ by letting

$$M(\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

a. Let $n = 4$, let $\sigma = (1\ 2\ 3)(4)$ and let $\tau = (1\ 2)(3\ 4)$. Write down $M(\sigma)$, $M(\tau)$, and $M(\sigma \circ \tau)$.
b. Show that $M(\sigma \circ \tau) = M(\sigma) \cdot M(\tau)$ for all permutations σ and τ of $[n]$ (not just those in part a).