Homework 8 : Due Wednesday, March 7

Problem 1: Let $m, n, k \in \mathbb{N}^+$ with $k \leq m$ and $k \leq n$. Calculate the number of sequences of zeros and ones of length $m + n$ which have both of the following properties:

- There are exactly m zeros and n ones.
- There are exactly k runs of ones.

Thus, you should no longer assume that the sequence starts with a one and ends with a zero.

Problem 2: Let

$$S = \{a \in [504] : \gcd(a, 504) = 1\}$$

That is, S consists of all numbers in [504] that are relatively prime to 504. Calculate $|S|$.
Hint: If $\gcd(a, 504) \neq 1$, then a and 504 share a common prime divisor.

Problem 3: In several cards games (bridge, spades, hearts, etc.) each player receives a 13-card hand from a standard 52-card deck.
a. How many such 13-card hands have at least one card of every suit? What percentage of all possible 13-card hands is this?
b. How many such 13-card hands have all four cards of some rank (e.g. all four queens)?

Problem 4: Determine the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 17$$

where each $x_i \in \mathbb{N}$ and each $x_i \leq 6$.

Problem 5: Suppose that you have n distinct letters. How many "words" of length $2n$ can you make with both of the following properties:

- Each letter is used exactly twice.
- No two consecutive letters agree.

Hint: Fix a given letter, say a. To count the number of "words" in which the two a's are consecutive, think about gluing the two a's together and treating them as one object.