Homework 7 : Due Friday, March 2

Problem 1: Let $n, k \in \mathbb{N}^+$. Count the number of solutions to

$$x_1 + x_2 + \cdots + x_k \leq n$$

where each $x_i \in \mathbb{N}$. For example, if $n = 2$ and $k = 2$, then there are 6 solutions given by the following ordered pairs (x_1, x_2):

$$(0,0) \quad (0,1) \quad (1,0) \quad (0,2) \quad (1,1) \quad (2,0)$$

Your final answer should not involve any summations.

Problem 2: Suppose that you have 12 identical apples and 1 orange.
a. In how many ways can you distribute the fruit to 4 distinct people?
b. In how many ways can you distribute the fruit to 4 distinct people in such a way that each person receives at least one piece of fruit?

Problem 3: Given a finite sequence of zeros and ones, define a *run* of ones to be a maximal consecutive subsequence of ones. For example, the sequence 11101100000100101100 has 5 runs of ones (and also 5 runs of zeros). Let $m, n, k \in \mathbb{N}^+$ with $k \leq m$ and $k \leq n$. Calculate the number of sequences of zeros and ones of length $m + n$ which have all three of the following properties:

- Have exactly m zeros and n ones.
- Starts with a one and ends with a zero.
- Have exactly k runs of ones.

Hint: How many runs of zeros must such a sequence have? Think about the lengths of the various runs and how they relate to compositions.

Problem 4: Show that $S(n, 2) = 2^{n-1} - 1$ for all $n \geq 2$ in the following two ways:
a. By induction.
b. By a combinatorial argument.

Problem 5: Recall that $S(n, n-1) = \binom{n}{2}$ for all $n \geq 2$. Show that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.

Problem 6: Let $n, k \in \mathbb{N}$ with $k < n$. Recall that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$$

which says that you can get an element of Pascal's triangle by adding up the elements which are simultaneously above it and one column to the left. Show that

$$S(n+1, k+1) = \binom{n}{0} \cdot S(n, k) + \binom{n}{1} \cdot S(n-1, k) + \binom{n}{2} \cdot S(n-2, k) + \cdots + \binom{n}{n-k} \cdot S(k, k)$$

Hint: Give a combinatorial argument. Think about the block containing $n + 1$.