Homework 6 : Due Monday, February 27

Problem 1: Recall that a *flush* in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are $5,148$ many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a "wild card". That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2's are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2's and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Problem 2:
a. Use the Binomial Theorem to show that $(1+x)^n \geq 1+nx$ whenever $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq 0$.
b. Use induction to show that $(1+x)^n \geq 1+nx$ whenever $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq -1$. Clearly explain where you are using the assumption that $x \geq -1$.

Problem 3: Let $n \in \mathbb{N}^+$.
a. Determine (with explanation) the value of

$$\sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n}$$

b. Determine (with explanation) the value of

$$\sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n}$$

c. Determine (with explanation) the value of

$$\sum_{k=2}^{n} k \cdot (k-1) \cdot \binom{n}{k} = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n-1) \cdot \binom{n}{n}$$

Problem 4: For any $k, n \in \mathbb{N}^+$ with $k \leq n$, we know that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ since each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people.
a. Let $k, m, n \in \mathbb{N}^+$ with $m \leq k \leq n$. Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$$

This generalizes the above result (the above is the special case where $m = 1$).
b. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Find a simple formula for:

$$\sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m}$$

Problem 5: Let $n \in \mathbb{N}^+$. We know from class that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Notice the right-hand side is simply $\binom{n+1}{2}$. Give a direct combinatorial proof that

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$$

by arguing that each side counts the number of ordered pairs (k, ℓ) with $k, \ell \in \mathbb{N}$ and $0 \leq k < \ell \leq n$.