Homework 4 : Due Monday, February 13

Problem 1: Let A, B, C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Show each of the following.
a. If $g \circ f$ is surjective, then g is surjective.
b. If $g \circ f$ is injective, then f is injective.
c. If $g \circ f$ is injective and f is surjective, then g is injective.
Note: Recall that to prove that f is injective, you should start by taking two arbitrary $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$ and try to deduce that $a_1 = a_2$. Also, to prove that g is surjective, start by taking and arbitrary $c \in C$, and show how to find $b \in B$ with $g(b) = c$.

Problem 2: Let $n \in \mathbb{N}^+$. Given any $n + 2$ many natural numbers, show that it always possible to find two of them whose sum or difference is divisible by $2n$.

Problem 3: A *lattice point* in the plane is a point of the form (a, b) where $a, b \in \mathbb{Z}$. For example, $(3, 5)$ is a lattice point but $(\pi, 1)$ is not. Show that given any 5 lattice points in the plane, there exists two of the points whose midpoint is also a lattice point.
Hint: Think about evens and odds.

Problem 4:
a. For each $n \in \mathbb{N}^+$, give an example of a set $S \subseteq [2n]$ with $|S| = n$ such that $\gcd(a, b) > 1$ for all $a, b \in S$ with $a \neq b$.
b. Let $n \in \mathbb{N}^+$. Suppose that $A \subseteq [2n]$ and $|A| = n + 1$. Show that there exists $a, b \in A$ with $a \neq b$ such that $\gcd(a, b) = 1$.

Problem 5: Suppose that you have a group of 10 people and that the age of every person in the group is between 1 and 100 (inclusive). Suppose also that all of the ages are distinct (so there are not two people of the same age).
a. Show that there exist two nonempty distinct subsets A and B of people such that the sum of the ages of the people in A equals the sum of the ages of the people in B.
b. Show moreover that you can find A and B as in part a which are also *disjoint*, i.e. for which no person is in both A and B.
Example: Suppose that the ages of the people in the group are $3, 7, 13, 19, 24, 30, 38, 49, 63, 78$. One such example is $A = \{3, 13, 78\}$ and $B = \{7, 19, 30, 38\}$ since $3 + 13 + 78 = 94 = 7 + 19 + 30 + 38$.
Hint: How many possible nonempty subsets of people are there? What's the largest possible sum?