Homework 3 : Due Wednesday, February 8

Note: Throughout this assignment, let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

Problem 1: A friend tries to convince you that $2 \mid f_n$ for all $n \geq 3$. Here is their argument using strong induction. For the base case, notice that $f_3 = 2$, so $2 \mid f_3$. For the inductive step, suppose that the $n \geq 4$ and we know the result for all k with $3 \leq k < n$. Since $f_n = f_{n-1} + f_{n-2}$ and we know by induction that $2 \mid f_{n-1}$ and $2 \mid f_{n-2}$, it follows that $2 \mid f_n$. Therefore, $2 \mid f_n$ for all $n \geq 3$.

Now you know that your friend's argument must be wrong because $f_7 = 13$ and $2 \nmid 13$. Pinpoint the fundamental error. Be as explicit and descriptive as you can.

Problem 2: We showed in class that $f_n \leq 2^n$ for all $n \in \mathbb{N}$ and we quickly discussed how this bound could be improved. We do that here. Let $\phi = \frac{1+\sqrt{5}}{2}$.
a. Show that $\phi^2 = \phi + 1$.
b. Show that $f_n \leq \phi^{n-1}$ for all $n \in \mathbb{N}$.
c. Show that $f_n \geq \phi^{n-2}$ for all $n \in \mathbb{N}^+$.

Problem 3: Let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Show that

$$f_{n+1} f_{n-1} = f_n^2 + (-1)^n$$

for all $n \in \mathbb{N}^+$.

Problem 4: Define a sequence recursively by letting $a_0 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$.
a. Show that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.
b. Show that the sequence is increasing, i.e. that $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
Cultural Aside: In analysis, you'll show that any increasing sequence which is bounded above converges to a limit. In this case, the limit turns out to equal $\phi = \frac{1+\sqrt{5}}{2}$.

Problem 5: Define a sequence a_n recursively by letting $a_0 = 0$, $a_1 = 1$, and

$$a_n = 3a_{n-1} - 2a_{n-2}$$

for $n \geq 2$. Show that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Problem 6: Use the Euclidean Algorithm to find the greatest common divisor of the following pairs of numbers a and b.

- $a = 234$ and $b = 165$
- $a = 562$ and $b = 471$

Furthermore, once you find the greatest common divisor d, find $k, \ell \in \mathbb{Z}$ such that $ak + b\ell = d$.