Homework 14 : Due Wednesday, April 25

Problem 1:
a. Let $n \geq 2$. Let G be a simple graph on n vertices with at least n edges. Show that G contains a cycle.
b. Show that if G is a simple graph on $n \geq 5$ vertices, then at least one of G or $\overline{G}$ has a cycle.

Problem 2: Recall the graph Q_n from Problem 2 on Homework 11. Show that Q_n is bipartite.

Problem 3: Let G be the following simple graph. Let V_G be the set of subsets of [5] of size 2. For the edge set E, include an edge with endpoints u and w exactly when u and w are disjoint subsets of [5] (i.e. exactly when they have no elements in common). Determine $\chi(G)$.

Problem 4: Given $n \in \mathbb{N}$ with $n \geq 3$, let C_n be the simple graph with vertex set $[n]$ where 1 and n are adjacent and also k and $k+1$ are adjacent whenever $1 \leq k \leq n-1$ (so C_n is just a cycle of length n). Determine the smallest possible maximal matching in C_{3k} for each $k \in \mathbb{N}^+$.

Problem 5: Show that a tree has at most one perfect matching.
Hint: Given two perfect matchings, think about the symmetric difference.

Problem 6: Let T be a tree.
a. Show that T is bipartite.
b. Show that given any valid 2-coloring of the vertices of T, there exists a leaf in the larger of the two color classes (and a leaf in each if both colors are used the same number of times).
Hint for b: Think about the sum of the degrees in each color class.