Homework 12 : Due Monday, April 16

Problem 1: Let G be a simple graph. Define a new simple graph $\overline{G}$, called the complement of G, as follows. Let $V_{\overline{G}} = V_G$, i.e. the vertex set of $\overline{G}$ is the vertex set of G. Given two distinct vertices u and w, include an edge in $\overline{G}$ with endpoints u and w exactly when no such edge exists in G.

Show that if G is a simple disconnected graph, then $\overline{G}$ is connected.

Problem 2: Let T be a finite tree with n vertices. Let a be the average degree of the vertices (i.e. the result of summing the degrees of the vertices and dividing by n).
a. Show that $a < 2$.
b. Show that if T has a vertex of degree ℓ, then T has at least ℓ leaves.

Problem 3: For each of the following, either prove or find a counterexample.
a. Deleting a vertex of maximum degree in a finite graph G cannot increase the average degree.
b. Deleting a vertex of minimum degree in a finite graph G cannot decrease the average degree.

Problem 4: A saturated hydrocarbon is a molecule C_kH_ℓ in which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds forms a cycle. Show that $\ell = 2k + 2$ in any saturated hydrocarbon.
Hint: Form a graph and determine the sum of the degrees.

Problem 5: Let T be a finite tree with at least two vertices in which $d(v) \geq 3$ whenever v is adjacent to a leaf. Show that there exist two leaves u and w of T that share a common neighbor.
Hint: Start by considering a longest possible path in G.

Problem 6: Let G be a finite simple graph with the property that $d(v) \geq 3$ for all $v \in V$. Prove that G has a cycle of even length.
Hint: See the hint for problem 5.