Homework 11 : Due Wednesday, April 11

Problem 1: Determine the maximum number of edges that a simple graph on n vertices can have. Explain.

Problem 2: Fix $n \in \mathbb{N}^+$. Consider the simple graph Q_n defined as follows. Let the vertex set V be the set of all sequences of 0's and 1's of length n (so for example, if $n = 3$, then one vertex is 010 and another is 111). For the edge set E, given $u, v \in V$, include an edge with endpoints u and v when they differ in exactly one coordinate (so for example when $n = 3$ there is edge with endpoints 001 and 101). The simple graph Q_n is called is called the n-cube.
a. Determine the number of vertices and edges in Q_n
b. Show that Q_n is connected.

Problem 3: Fix $n \in \mathbb{N}^+$. Consider the simple graph G_n defined as follows. As above, let the vertex set V be the set of all sequences of 0's and 1's of length n. For the edge set E, given $u, v \in V$, include an edge with endpoints u and v when they differ in exactly two coordinates.
a. Determine the number of edges in G_n.
b. Determine the number of connected components in G_n.

Problem 4: Let G be a graph with n vertices and k edges. Let δ be the minimum degree of any vertex in G, and let Δ be the maximum degree of any vertex in G. Show that

$$\delta \leq \frac{2k}{n} \leq \Delta$$

Problem 5: Suppose that G is a graph with exactly two vertices u and w of odd degree. Show that there is a u, w-path in G.

Problem 6: Let G be a connected graph and let $v \in V$. Show that v has a neighbor in each connected component of $G - v$ (where $G - v$ is the graph obtained by deleting v and all edges incident to v).

Problem 7: Let G be a simple graph with n vertices. Suppose that no vertex of G is isolated (i.e. no vertex has degree 0) and that no induced subgraph of G has exactly two edges. Show that G is a complete simple graph, i.e. show that every pair of distinct vertices of G are adjacent.