

## Homework 10 : Due Friday, April 6

**Problem 1:** Let  $A$  and  $B$  be two countably infinite sets. Show that  $A \cup B$  is countably infinite.

**Problem 2:** Let  $S$  be the set of all infinite sequences of 0's and 1's (so an element of  $S$  looks like 11000101110...). Show that  $S$  is uncountable.

**Problem 3:** Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.

a.  $A = \mathbb{Z}$  where  $a \sim b$  means  $a - b \neq 1$ .

b.  $A = \mathbb{Z}$  where  $a \sim b$  means that both  $a$  and  $b$  are even.

c. Let  $A$  be the set of all subsets of  $[5]$  (so some elements of  $A$  are  $\{5\}$ ,  $\{1, 3, 4\}$ , etc.). Given  $X, Y \in A$ , define  $X \sim Y$  to mean that  $X \cap Y \neq \emptyset$ , i.e. that  $X$  and  $Y$  have at least one element in common.

d. Let  $A$  be the set of all  $2 \times 2$  matrices with entries in  $\mathbb{R}$ . Given  $M, N \in A$  define  $M \sim N$  to mean that there exists an invertible  $2 \times 2$  matrix  $P$  with  $M = PNP^{-1}$ .

**Problem 4:** Fix  $n \in \mathbb{N}^+$ . Define a relation  $\sim$  on  $\mathbb{Z}$  by letting  $a \sim b$  mean that  $n \mid (a - b)$ .

a. Show that  $\sim$  is an equivalence relation on  $\mathbb{Z}$ .

b. Show that  $a \sim b$  if and only if  $a$  and  $b$  leave the same unique remainder upon division by  $n$ . Note that you must prove both directions here because the statement is "if and only if".

c. How many distinct equivalence classes does  $\sim$  have? Explain.

**Problem 5:** Let  $A$  be the set  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , i.e.  $A$  is the set of points in the plane with the origin removed. Define a relation  $\sim$  on  $A$  by letting  $(x_1, y_1) \sim (x_2, y_2)$  mean that there is a real number  $\lambda \neq 0$  with  $(x_1, y_1) = (\lambda x_2, \lambda y_2)$ . Let

$$S = \{(0, 1)\} \cup \{(1, m) : m \in \mathbb{R}\}$$

so  $S$  is the set of points on the line  $x = 1$  together with the point  $(0, 1)$ .

a. Show that  $\sim$  is an equivalence relation on  $A$ .

b. Show that every element of  $A$  is equivalent to some element of  $S$ .

c. Show that any two distinct elements of  $S$  are not equivalent.

**Problem 6:** A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

"If  $a \sim b$ , then  $b \sim a$  by symmetry, so  $a \sim a$  by transitivity. This gives the reflexive property."

Now you know that their argument must be wrong because one of the examples in Problem 3 is symmetric and transitive but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.