Homework 1 : Due Monday, January 30

Problem 1: Let $a, b, c \in \mathbb{Z}$. For each of the following, either prove or find a counterexample.
a. If $ab \mid c$, then $a \mid c$ and $b \mid c$.
b. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.
c. If $a \mid c$ and $b \mid c$, then $ab \mid c$.
d. If $a \mid bc$, then $a \mid b$ or $a \mid c$.
Note: In mathematics, when we say "P or Q", we include the possibility that both P and Q are true. Thus, "or" means "inclusive or", not "exclusive or".

Problem 2: Let $a, b, c \in \mathbb{Z}$. Consider the statement "If $ac \mid bc$, then $a \mid b$".
a. As given, the statement is false. Show this by providing an explicit counterexample.
b. Adjust the statement so that it is true. Give a proof that your adjusted statement is correct.

Problem 3: Let $a, b, c \in \mathbb{Z}$.
a. Suppose that $a \nmid b$ and $a \nmid c$. Show that it is possible that $a \mid (b + c)$.
b. Suppose that $a \nmid b$ and $a \nmid c$. Show that it is possible that $a \nmid (b + c)$.
c. Suppose that $a \mid b$ and $a \nmid c$. Show that $a \nmid (b + c)$.

Problem 4: Show that $3 \mid [(n + 1)^3 - (n^3 + 1)]$ for all $n \in \mathbb{Z}$.

Problem 5: Show that there do not exist $a, b \in \mathbb{Z}$ with $35a^2 + 14b^3 - 4 = 0$.

Problem 6: Show that if $4 \mid a$, then a is the difference of two perfect squares, i.e. there exist $b, c \in \mathbb{Z}$ with $a = b^2 - c^2$.