Homework 9: Due Friday, October 14

Problem 1: Recall that a *flush* in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are $5,148$ many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a "wild card". That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2's are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2's and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Problem 2:
a. Let $n \in \mathbb{N}^+$ and let $x \in \mathbb{R}$ with $x \geq 0$. Use the Binomial Theorem to show that $(1+x)^n \geq 1 + nx$.
b. Show that

$$1 \leq \sqrt[n]{2} \leq 1 + \frac{1}{n}$$

for all $n \in \mathbb{N}^+$.
Cultural Aside: Using the Squeeze Theorem, it follows that $\lim_{n \to \infty} \sqrt[n]{2} = 1$.

Problem 3: Let $n \in \mathbb{N}^+$. Determine (with explanation), the value of each of the following sums:
a.

$$\sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n}.$$

b.

$$\sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n}.$$

c.

$$\sum_{k=2}^{n} k \cdot (k-1) \cdot \binom{n}{k} = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n-1) \cdot \binom{n}{n}.$$

Problem 4: For all $k, n \in \mathbb{N}^+$ with $k \leq n$, we know that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ since each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people.
a. Let $k, m, n \in \mathbb{N}^+$ with $m \leq k \leq n$. Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}.$$

This generalizes the above result (which is the special case where $m = 1$).
b. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Find a simple formula for:

$$\sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m}.$$

Problem 5: Let a_n be the number of subsets of $[n] = \{1, 2, 3, \ldots, n\}$ that do *not* have two consecutive numbers (so when $n = 5$, we allow $\{1, 4\}$ but we do not allow $\{1, 4, 5\}$). Notice that:

- $a_0 = 1$ because $\emptyset$ is the only possibility.
- $a_1 = 2$ because $\emptyset$ and $\{1\}$ are both included.
- $a_2 = 3$ because $\emptyset$, $\{1\}$, and $\{2\}$ are all included, but $\{1, 2\}$ is not.

a. Give a combinatorial proof that $a_n = a_{n-1} + a_{n-2}$ whenever $n \geq 2$.
b. Using part a, explain why $a_n = f_{n+2}$ for all $n \in \mathbb{N}$, where f_k is the k^{th} Fibonacci number.