Homework 16: Due Wednesday, December 7

Problem 1: In this problem, you will finish the proof of Proposition 6.6.3. Let $m = pq$ where $p, q \in \mathbb{N}^+$ are distinct primes. Let $k, \ell \in \mathbb{N}^+$ with $k\ell \equiv 1 \pmod{\varphi(m)}$. Show that $(a^k)^\ell \equiv a \pmod{m}$ for all $a \in \mathbb{Z}$.
Hint: It suffices to show that both $(a^k)^\ell \equiv a \pmod{p}$ and $(a^k)^\ell \equiv a \pmod{q}$.

Problem 2: Show that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

for all $n \in \mathbb{N}^+$ by the interpreting the left-hand side as a certain Riemann sum.
Note: You can find a direct inductive proof of this fact in the proof of Proposition 2.1.8.

Problem 3: Let $n, k \in \mathbb{N}$ with $0 \leq k < n$.
a. Show that if $0 \leq k < \frac{n-1}{2}$, then $\binom{n}{k} < \binom{n}{k+1}$.
b. Show that if $\frac{n-1}{2} < k < n$, then $\binom{n}{k} > \binom{n}{k+1}$.
c. Show that if $k = \frac{n-1}{2}$, then $\binom{n}{k} = \binom{n}{k+1}$.
Note: This problem says that any row of Pascal's triangle is strictly increasing until the middle term(s), and then strictly decreasing afterwards. I suggest that you start by looking at the quotient $\binom{n}{k+1}/\binom{n}{k}$.

Problem 4: Show that there are arbitrarily large gaps in the primes, i.e. that for every $n \in \mathbb{N}^+$, there exist n consecutive composite numbers.
Hint: Factorials are your friends.

Interlude: A natural number $n \in \mathbb{N}^+$ is called *square-free* if it is not divisible by the square of any natural number greater than or equal to 2. It is straightforward to show that n is square-free if and only if it not divisible by p^2 for every prime p. Using the Fundamental Theorem of Arithmetic, it follows that n is square-free if and only if either $n = 1$, or n is the product of distinct primes.

Problem 5: Given $n \in \mathbb{N}^+$, show that $|\{k \in [n] : k \text{ is square-free}\}| \leq 2^{\pi(n)}$.

Problem 6: It is possible to show that for all $n \in \mathbb{N}^+$, there exist unique $k, m \in \mathbb{N}^+$ such that $n = k^2 m$ and such that m is square-free (essentially you factor n into primes, and form m by pulling out one of each prime that occurs an odd number of times). Use this result and the previous problem to show that

$$\pi(n) \geq \frac{\ln n}{2 \ln 2}$$

for all $n \in \mathbb{N}^+$.