Homework 14: Due Friday, November 18

Problem 1: Let $a, b, c \in \mathbb{Z}$. Show that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.

Problem 2: Let $m \in \mathbb{Z}$ with $m \equiv 3 \pmod 4$. Show that there does not exist $a, b \in \mathbb{Z}$ with $m = a^2 + b^2$.
Hint: Start by consider the possible values of a^2 modulo 4.

Problem 3: Show that $19 \nmid 4n^2 + 4$ for all $n \in \mathbb{Z}$.

Problem 4: Find, with full explanation, the remainder when dividing 3^{846} by 308.

Problem 5: Show that $\varphi(n)$ is even whenever $n > 2$.

Problem 6: Wilson's Theorem says that $(p-1)! \equiv -1 \pmod p$ whenever p is prime. Notice that $(4-1)! = 6$, so $(4-1)! \equiv 2 \pmod 4$. Show that if $n \in \mathbb{N}$ is composite and $n > 4$, then $(n-1)! \equiv 0 \pmod n$.
Note: In particular, it follows that $(n-1)! \not\equiv -1 \pmod n$ whenever n is composite, giving a converse to Wilson's Theorem.