Homework 12: Due Wednesday, November 9

Problem 1: Let $n \in \mathbb{N}^+$.
a. Evaluate
$$\sum_{k=0}^{n} 3^k \cdot c(n,k).$$
b. Evaluate
$$\sum_{k=0}^{n} 3^k \cdot s(n,k).$$
Note: Simplify your answers as much as possible.

Problem 2: Let $n \in \mathbb{N}$ with $n \geq 2$.
a. Show that
$$c(n,2) = \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{k(n-k)}.$$
b. Show that $c(n,2) = (n-1)! \cdot H_{n-1}$, where H_{n-1} is as defined in the interlude on Homework 11.

Problem 3: Show that for all $a \in \mathbb{Z}$, either $a^2 \equiv 0 \pmod 3$ or $a^2 \equiv 1 \pmod 3$.
Note: This problem is equivalent to Problem 1 on Homework 4. However, you should *not* just appeal to Problem 1 on Homework 4. Instead, use properties of congruences.

Problem 4: Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}^+$ be such that $a \equiv b \pmod m$. Show that $\gcd(a,m) = \gcd(b,m)$.

Problem 5: Suppose that $m, k \in \mathbb{N}^+$ and $a, b \in \mathbb{Z}$ are such that $ka \equiv kb \pmod m$. Let $d = \gcd(k,m)$, and fix $n \in \mathbb{N}^+$ with $m = dn$. Show that $a \equiv b \pmod n$.

Problem 6: Let $p \in \mathbb{N}^+$ be prime and let $a \in \mathbb{Z}$. Show that $a^2 \equiv 1 \pmod p$ if and only if either $a \equiv 1 \pmod p$ or $a \equiv -1 \pmod p$.

Problem 7: Let $p \in \mathbb{N}^+$ be prime. Define a function $ord_p \colon \mathbb{N}^+ \to \mathbb{N}$ as follows. Given $a \in \mathbb{N}^+$, let $ord_p(a)$ be the largest $k \in \mathbb{N}$ such that $p^k \mid a$. For example, we have $ord_3(45) = 2$ and $ord_3(10) = 0$. Without using the Fundamental Theorem of Arithmetic, prove that for all $p, a, b \in \mathbb{N}^+$ with p prime, we have $ord_p(ab) = ord_p(a) + ord_p(b)$.
Note: Think carefully about what you need to do in order to prove that $ord_p(c) = k$. You need to show that $p^k \mid c$, but you also need to show that $p^\ell \nmid c$ whenever $\ell > k$.