Homework 11: Due Friday, November 4

Problem 1: Consider all 10^{10} many ten digit numbers where you allow leading zeros (so 0018345089 is one possibility). How many such numbers have the property that every odd digit occurs at least once?

Problem 2: Determine the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where each $x_i \in \mathbb{N}$ and each $x_i \leq 6$.

Problem 3: Let $m, n \in \mathbb{N}^+$ with $m \leq n$. If σ is a permutation of $[n]$, then we say that $i \in [n]$ is a fixed point of σ if $\sigma(i) = i$. How many permutations of $[n]$ have exactly m fixed points?

Problem 4: On Homework 10, you showed that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$. Now show that

$$c(n, n-2) = 2 \cdot \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.

Problem 5: Let $n \in \mathbb{N}^+$.
a. How many ways are there to break up $3n$ (distinct) people into n groups of size 3, where there is no ordering amongst the groups (so all that matters is the people who are grouped together)?
b. How many permutations of $[3n]$ consist of n disjoint 3-cycles?
Note: Simplify your answers as much as possible.

Interlude: Given $n \in \mathbb{N}^+$, define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. The rational numbers H_n are called *harmonic numbers*. It can be shown that as n gets large, then H_n gets arbitrary large, and in fact H_n is reasonably close to $\ln n$ (roughly, H_n can viewed as a certain Riemann sum approximating $\int_1^n \frac{1}{x}\, dx$, which is where the $\ln n$ comes from). More precisely, it can be shown that

$$\lim_{n \to \infty} (H_n - \ln n)$$

exists and equals a number $\gamma \approx .5772156649\ldots$ called the Euler-Mascheroni constant (remarkably, it is still not known whether γ is irrational). Thus, when n is large, an extremely good approximation to H_n is

$$H_n \approx \ln n + \gamma.$$

Problem 6: Let $n \in \mathbb{N}^+$.
a. Suppose that $k \in \mathbb{N}^+$ with $n + 1 \leq k \leq 2n$. Show that the number of permutations of $[2n]$ containing a k-cycle is $\frac{(2n)!}{k}$. Explicitly describe where and how you are using the assumption that $k \geq n + 1$.
b. Use part a and the above discussion to show that when n is large, the fraction of permutations of $[2n]$ containing a cycle of length at least $n + 1$ is approximately $\ln 2$.