# Homework 10: Due Monday, October 31

**Problem 1:** Let $n, k \in \mathbb{N}^+$. Count the number of solutions to

$$x_1 + x_2 + \cdots + x_k \leq n$$

where each $x_i \in \mathbb{N}$. For example, if $n = 2$ and $k = 2$, then there are 6 solutions given by the following ordered pairs $(x_1, x_2)$:

$$(0,0) \quad (0,1) \quad (1,0) \quad (0,2) \quad (1,1) \quad (2,0).$$

Your final answer should not involve any summations.

**Problem 2:** Suppose that you have 12 identical apples and 1 orange.
a. In how many ways can you distribute the fruit to 4 distinct people?
b. In how many ways can you distribute the fruit to 4 distinct people in such a way that each person receives at least one piece of fruit?

**Problem 3:** Let $B(n, k)$ be the number of compositions of $n$ into $k$ parts. Using a direct combinatorial argument (so without using our formula), show that

$$B(n, k) = B(n-1, k-1) + B(n-1, k)$$

for all $n, k \in \mathbb{N}$ with $2 \leq k \leq n$.

**Problem 4:** Show that $S(n, 2) = 2^{n-1} - 1$ for all $n \geq 2$ in the following two ways:
a. By induction.
b. By a combinatorial argument.

**Problem 5:** Recall that $S(n, n-1) = \binom{n}{2}$ for all $n \geq 2$. Show that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.

**Problem 6:** In several cards games (bridge, spades, hearts, etc.) each player receives a 13-card hand from a standard 52-card deck.
a. How many such 13-card hands have at least one card of every suit? What percentage of all possible 13-card hands is this?
b. How many such 13-card hands have all four cards of some rank (e.g. all four queens)?