Homework 9 : Due Friday, October 28

Given $n \in \mathbb{N}^+$, define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. The rational numbers H_n are called *harmonic numbers*. It can be shown that as n gets large, then H_n gets arbitrary large and in fact H_n is reasonably close to $\ln n$ (roughly, H_n can viewed as a certain Riemann sum approximating $\int_1^n \frac{1}{x}\,dx$, which is where the $\ln n$ comes from). More precisely, it can be shown that

$$\lim_{n\to\infty} (H_n - \ln n)$$

exists and equals a number $\gamma \approx .5772156649\ldots$ called the Euler-Mascheroni constant (remarkably, it is still not known whether γ is irrational). Thus, when n is large, an extemely good approximation to H_n is

$$H_n \approx \ln n + \gamma$$

Problem 1: Let $n \in \mathbb{N}$ with $n \geq 2$.
a. Show that

$$c(n,2) = \frac{n!}{2} \sum_{k=1}^{n-1} \frac{1}{k(n-k)}$$

Be careful in the case when n is even!
b. Show that $c(n,2) = (n-1)! \cdot H_{n-1}$.

Problem 2: Let $n \in \mathbb{N}^+$.
a. Suppose that $k \in \mathbb{N}^+$ with $n+1 \leq k \leq 2n$. Show that the number of permutations of $[2n]$ containing a k-cycle is $\frac{(2n)!}{k}$.
b. Use part a and the above discussion to show that when n is large, the fraction of permutations of $[2n]$ containing a cycle of length at least $n+1$ is approximately $\ln 2$.

Problem 3: In several cards games (bridge, spades, hearts, etc.) each player receives a 13-card hand from a standard 52-card deck.
a. How many such 13-card hands have at least one card of every suit? What percentage of all possible 13-card hands is this?
b. How many such 13-card hands have all four cards of some rank (e.g. all four queens)?

Problem 4: Consider all 10^{10} many ten digit numbers where you allow leading zeros (so 0018345089 is one possibility). How many such numbers have the property that every odd digit occurs at least once?

Problem 5: Suppose that you have n distinct letters. How many "words" of length $2n$ can you make with both of the following properties:

- Each letter is used exactly twice.
- No two consecutive letters agree.

Hint: Fix a given letter, say a. To count the number of "words" in which the two a's are consecutive, think about gluing the two a's together and treating them as one object.