Homework 8 : Due Friday, October 14

Problem 1: Let $m, n, k \in \mathbb{N}^+$ with $k \leq m$ and $k \leq n$. Calculate the number of sequences of zeros and ones of length $m + n$ which have both of the following properties:

- There are exactly m zeros and n ones.
- There are exactly k runs of ones.

Thus, you should no longer assume that the sequence starts with a one and ends with a zero.

Problem 2: Recall that $p(n)$ is the number of partitions of n, and that $p(n, k) = p_k(n)$ is the number of partitions of n into exactly k parts.
a. Let $n \in \mathbb{N}^+$. Show that

$$p(n, 2) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

b. Let $n \in \mathbb{N}^+$. Show that $p(2n + 1, n + 1) = p(n)$.
c. Compute $p(15, 8)$. You may use the table of $p(n, k)$ for $1 \leq n \leq 7$ and $1 \leq k \leq 7$ from class.

Problem 3:
a. Let $n \in \mathbb{N}^+$. Show that $p(n + 1) - p(n)$ is the number of partitions of $n + 1$ into parts each of which has size at least 2.
b. Let $n \in \mathbb{N}^+$. Show that $p(n + 2) + p(n) \geq 2 \cdot p(n + 1)$.

Problem 4: Let $n \in \mathbb{N}^+$.
a. Show that

$$\sum_{k=0}^{n} c(n, k) = n!$$

b. Show that

$$\sum_{k=0}^{n} (-1)^k c(n, k) = 0$$

whenever $n \geq 2$.

Problem 5: On the last homework, you showed that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$. Now show that

$$c(n, n-2) = 2 \cdot \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.