Homework 7 : Due Monday, October 10

Problem 1: Let $n, k \in \mathbb{N}^+$. Count the number of solutions to

$$x_1 + x_2 + \cdots + x_k \leq n$$

where each $x_i \in \mathbb{N}$. For example, if $n = 2$ and $k = 2$, then there are 6 solutions given by the following ordered pairs (x_1, x_2):

$$(0,0) \quad (0,1) \quad (1,0) \quad (0,2) \quad (1,1) \quad (2,0)$$

Your answer should not involve any summations.

Problem 2: Given a finite sequence of zeros and ones, define a *run* of ones to be a maximal consecutive subsequence of ones. For example, the sequence 11101100000100101100 has 5 runs of ones (and also 5 runs of zeros). Let $m, n, k \in \mathbb{N}^+$ with $k \leq m$ and $k \leq n$. Calculate the number of sequence of zeros and ones of length $m + n$ which:

- Have exactly m zeros and n ones.
- Start with a one and end with a zero.
- Have exactly k runs of ones.

Hint: How many runs of zeros must such a sequence have? Then think about compositions.

Problem 3: Show that $S(n, 2) = 2^{n-1} - 1$ for all $n \geq 2$ in the following three ways:
a. By induction.
b. By counting the number of surjections $f\colon [n] \to [2]$, and using that to determine $S(n, 2)$.
c. By a direct combinatorial argument.
Hint for b: It's easier to count the number of functions $f\colon [n] \to [2]$ which are *not* surjective.

Problem 4: Recall that $S(n, n-1) = \binom{n}{2}$ for all $n \geq 2$. Show that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$ in the following two ways:
a. By induction.
b. By a combinatorial argument (with a bit of algebra if necessary).

Problem 5: Let $n, k \in \mathbb{N}$ with $k < n$. Recall that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$$

which says that you can get an element of Pascal's triangle by adding up the elements which are simultaneously above it and one column to the left. In this problem, we give two distinct analogues for $S(n, k)$.
a. Show that

$$S(n+1, k+1) = 1 \cdot S(n, k) + (k+1) \cdot S(n-1, k) + (k+1)^2 \cdot S(n-2, k) + \cdots + (k+1)^{n-k} \cdot S(k, k)$$

b. Show that

$$S(n+1, k+1) = \binom{n}{0} \cdot S(n, k) + \binom{n}{1} \cdot S(n-1, k) + \binom{n}{2} \cdot S(n-2, k) + \cdots + \binom{n}{n-k} \cdot S(k, k)$$