Homework 6 : Due Wednesday, October 5

Problem 1: Let $n \in \mathbb{N}^+$. Determine the value of

$$\sum_{k=0}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n}$$

Problem 2: Let $n \in \mathbb{N}^+$. Find a simple formula for

$$\sum_{k=0}^{n} k^2 \cdot \binom{n}{k}$$

Problem 3: Let $n \in \mathbb{N}^+$. Show that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Problem 4: For any $k, n \in \mathbb{N}^+$ with $k \leq n$, we know that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ since each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people.
a. Let $k, m, n \in \mathbb{N}^+$ with $m \leq k \leq n$. Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$$

This generalizes the above result (the above is the special case where $m = 1$).
b. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Find a simple formula for:

$$\sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m}$$

Problem 5: In class, we derived the formula

$$1^2 + 2^2 + \cdots + n^2 = 2 \cdot \binom{n+1}{3} + \binom{n+1}{2}$$

using some algebra and facts about sums of binomial coefficients. Give a direct combinatorial proof of this by arguing that both sides count the number of triples (a, b, c) where $a, b, c \in \{0, 1, 2, \ldots, n\}$ and $c > \max\{a, b\}$.

Problem 6:
a. Find integers A, B, C such that

$$m^3 = A \cdot \binom{m}{3} + B \cdot \binom{m}{2} + C \cdot \binom{m}{1}$$

for all $m \in \mathbb{N}^+$.
b. Use part a to derive a formula for $1^3 + 2^3 + \cdots + n^3$ for each $n \in \mathbb{N}^+$.