Homework 5 : Due Friday, September 30

Note: For each of these counting problems, you must explain your solution. For example, if your answer is a product, describe the sequence of choices you are making and explain where each term comes from. Numerical answers without written justification will receive no credit.

Problem 1: A local pizza place has three different types of crust, five different meats, and seven different (non meat) toppings. For a given pizza, you can pick any of crusts, at most 2 meats (so 0, 1, or 2 is possible) and at most 3 toppings (so 0, 1, 2, or 3 is possible). How many pizzas are possible?

Problem 2: How many ways are there to pick two cards from a standard deck such that the first card is a spade and the second is not an ace? In this problem, order matters. So if you pick the 3 of spades followed by the 7 of spades, this is different from the 7 of spades followed by the 3 of spades.

Problem 3: Suppose that a lottery draws 6 numbers from $[60] = \{1, 2, \ldots, 60\}$ without replacement and where order drawn doesn't matter (so the result is a subset of $[60]$ of size 6). What percentage of possible lottery numbers have 3 evens and 3 odds?

Problem 4: Let $n > 1$. Suppose that you flip a coin $2n$ times to obtain a sequence of heads of tails.
a. How many such sequences have an equal number of heads and tails?
b. How many such sequences have the property that the number of heads and number of tails differ by 2 (in either direction)?
c. Which of the answers in part a and part b is larger? Explain.

Problem 5: Consider a round robin tournament with n teams, i.e. a tournament where each pair of teams plays exactly one game against each other. Suppose that every game ends in either a win or a loss for each team (i.e. ties are not possible). How many possible outcomes are there?

Problem 6: Suppose that you have an 8×8 chessboard. You want to place 8 rooks on the chessboard in such a way so that no two rooks can strike each other (i.e. that each row and each column contains exactly one rook).
a. How many ways can you do this if the rooks all have different colors?
b. How many ways can you do this if the rooks are identical?
c. How many ways can you do this if 4 rooks are red and 4 rooks are blue?

Problem 7: How many 5-card poker hands have at least one card of every suit?

Problem 8: Recall that a *flush* in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are $5,148$ many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a "wild card". That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2's are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2's and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Problem 9: We know from class that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Notice the right-hand side is simply $\binom{n+1}{2}$. Give a direct combinatorial proof that

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$$

by arguing that each side counts the number of ordered pairs (k, ℓ) with $0 \leq k < \ell \leq n$.