Homework 4 : Due Monday, September 19

Problem 1: In this problem, we show that $\sqrt{p}$ is irrational for all primes $p \in \mathbb{N}^+$. Suppose then that p is prime and assume, for the sake of obtaining a contradiction, that $\sqrt{p}$ is rational. We may then fix $a \in \mathbb{Z}$ and $b \in \mathbb{N}^+$ with $\sqrt{p} = \frac{a}{b}$. Furthermore, we may assume that $\gcd(a, b) = 1$ by dividing out common factors (or more precisely by dividing both a and b by $\gcd(a, b)$). Squaring both sides of the above equation we conclude that $p = \frac{a^2}{b^2}$ and hence $pb^2 = a^2$.
a. Show that $p \mid a$.
b. Show that $p \mid b$.
c. Explain how this gives a contradiction, and hence why $\sqrt{p}$ is irrational.

Problem 2: Let $n \in \mathbb{N}^+$. Suppose that $A \subseteq [3n]$ and $|A| = n + 1$, i.e. suppose that A is a set of $n + 1$ many elements of $[3n] = \{1, 2, \dots, 3n\}$. Show that there exists $a, b \in A$ with $a \neq b$ such that $|a - b| \leq 2$.

Problem 3: Let $n \in \mathbb{N}^+$.
a. Give an example of a set $S \subseteq [2n]$ with $|S| = n$ such that $\gcd(a, b) > 1$ for all $a, b \in S$ with $a \neq b$.
b. Suppose that $A \subseteq [2n]$ and $|A| = n+1$. Show that there exists $a, b \in A$ with $a \neq b$ such that $\gcd(a, b) = 1$.

Problem 4: Let A, B, C be sets. Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Show each of the following.
a. If $g \circ f$ is surjective, then g is surjective.
b. If $g \circ f$ is injective, then f is injective.
c. If $g \circ f$ is injective and f is surjective, then g is injective.

Problem 5: Suppose that you have a group of 10 people and that the age of every person in the group is between 1 and 100 (inclusive).
a. Show that there exist two nonempty distinct subsets A and B of people such that the sum of the ages of the people in A equals the sum of the ages of the people in B.
b. Show moreover that you can find A and B as in part a which are also *disjoint*, i.e. for which no person is in both A and B.
Example: Suppose that the ages of the people in the group are $3, 7, 13, 19, 24, 30, 38, 49, 63, 78$. One such example is $A = \{3, 13, 78\}$ and $B = \{7, 19, 30, 38\}$ since $3 + 13 + 78 = 94 = 7 + 19 + 30 + 38$.

Problem 6: Suppose that you have 5 positive real numbers which sum to 100. Show that there exist two of the numbers such that the distance between them is at most 10.
Hint: Write the five numbers in increasing order as $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. Start by considering the case when $a_5 \leq 40$. Once you handle that, suppose that $a_5 > 40$. Now think about a threshold value for a_4 that would allow you to solve the problem quickly.