Homework 3 : Due Monday, September 12

Note: Throughout this assignment, let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

Problem 1: A friend tries to convince you that $2 \mid f_n$ for all $n \geq 3$. Here is their argument using strong induction. For the base case, notice that $f_3 = 2$, so $2 \mid f_3$. For the inductive step, suppose that the $n \geq 4$ and we know the result for all k with $3 \leq k < n$. Since $f_n = f_{n-1} + f_{n-2}$ and we know by induction that $2 \mid f_{n-1}$ and $2 \mid f_{n-2}$, it follows that $2 \mid f_n$. Therefore, $2 \mid f_n$ for all $n \geq 3$.

Now you know that your friend's argument must be wrong because $f_7 = 13$ and $2 \nmid 13$. Pinpoint the fundamental error. Be as explicit and descriptive as you can.

Problem 2: Use the Euclidean Algorithm to find the greatest common divisor of the following pairs of numbers a and b.

- $a = 234$ and $b = 165$
- $a = 562$ and $b = 471$

Furthermore, once you find the greatest common divisor d, find $k, \ell \in \mathbb{Z}$ such that $ak + b\ell = d$.

Problem 3: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid c$, that $b \mid c$, and that $\gcd(a, b) = 1$. Show that $ab \mid c$.

Problem 4: Show that $\gcd(f_n, f_{n+1}) = 1$ for all $n \in \mathbb{N}$.

Problem 5: By definition, we have $f_{n+2} = f_{n+1} + f_n = 1 \cdot f_{n+1} + 1 \cdot f_n$. From here, we get

$$\begin{aligned} f_{n+3} &= f_{n+2} + f_{n+1} \\ &= (1 \cdot f_{n+1} + 1 \cdot f_n) + 1 \cdot f_{n+1} \\ &= 2 \cdot f_{n+1} + 1 \cdot f_n \end{aligned}$$

$$\begin{aligned} f_{n+4} &= f_{n+3} + f_{n+2} \\ &= (2 \cdot f_{n+1} + 1 \cdot f_n) + (1 \cdot f_{n+1} + 1 \cdot f_n) \\ &= 3 \cdot f_{n+1} + 2 \cdot f_n \end{aligned}$$

$$\begin{aligned} f_{n+5} &= f_{n+4} + f_{n+3} \\ &= (3 \cdot f_{n+1} + 2 \cdot f_n) + (2 \cdot f_{n+1} + 1 \cdot f_n) \\ &= 5 \cdot f_{n+1} + 3 \cdot f_n \end{aligned}$$

Thus, in each case, we can write f_{n+k} in terms of f_{n+1} and f_n. Furthermore, the coefficients that we get appear to be Fibonacci numbers! This is no accident. Show that

$$f_{n+k} = f_k \cdot f_{n+1} + f_{k-1} \cdot f_n$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N}^+$.

Problem 6: Show that if $d, n \in \mathbb{N}^+$ and $d \mid n$, then $f_d \mid f_n$.
Hint: You want to show that $f_d \mid f_{dm}$ for all $m \in \mathbb{N}^+$. Problem 5 is helpful.