Homework 2 : Due Wednesday, September 7

Problem 1: For each $n \in \mathbb{N}^+$, let $f_n(x)$ be the function $f_n(x) = x^n$. Use induction and the product rule for derivatives to show that $f_n'(x) = nx^{n-1}$ for each $n \in \mathbb{N}^+$.

Problem 2: Let $x \in \mathbb{R}$ with $x \geq 0$.
a. Show that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}^+$.
b. Show that

$$1 \leq \sqrt[n]{2} \leq 1 + \frac{1}{n}$$

for all $n \in \mathbb{N}^+$.
Cultural Aside: Using the Squeeze Theorem, it follows that $\lim\limits_{n\to\infty} \sqrt[n]{2} = 1$.

Problem 3: Show that if $n \in \mathbb{N}$ and $n \geq 10$, then $2^n > n^3$.

Problem 4: Define a sequence recursively by letting $a_0 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$.
a. Show that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.
b. Show that the sequence is increasing, i.e. that $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
Cultural Aside: In analysis, you'll show that any increasing sequence which is bounded above converges to a limit. In this case, the limit turns out to equal $\phi = \frac{1+\sqrt{5}}{2}$.

Problem 5: Let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Use (strong) induction to show that

$$f_{n+1}f_{n-1} = f_n^2 + (-1)^n$$

for all $n \in \mathbb{N}^+$.

Problem 6: Let f_n be as in Problem 5. Show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

for all $n \in \mathbb{N}^+$.
b. Use part a and some linear algebra to give another proof of Problem 5.