Homework 13 : Due Wednesday, November 23

Problem 1: Let $n \in \mathbb{N}^+$. Recall the graph Q_n from Problem 4 on Homework 10.
a. Show that Q_n is bipartite.
b. Explicitly describe a perfect matching of Q_n.

Problem 2: Let G be a graph. Let M be a maximum matching in G and suppose that M has k edges. Show that if L is a maximal matching in G, then L has at least $\frac{k}{2}$ many edges.

Problem 3: Let G be a bipartite graph. Suppose that every vertex in G has the same common degree $k \in \mathbb{N}^+$. Show that G has a perfect matching.

Problem 4: Fix a finite graph G. Consider the following game with two players. Player 1 starts by picking a vertex v_1 of G. Player 2 responds by picking a new vertex v_2 adjacent to v_1. Player 1 then picks a vertex $v_3 \notin \{v_1, v_2\}$ which is adjacent to v_2. This process continues so that at each stage, the corresponding player picks $v_{n+1} \notin \{v_1, v_2, \dots, v_n\}$ which is adjacent to v_n. The game ends when a player is unable to make a valid move, in which case the other player is declared the winner.
a. Suppose that G has a perfect matching. Explicitly describe a winning strategy for Player 2 and explain why it works.
b. Suppose that G does not have a perfect matching. Explicitly describe a winning strategy for Player 1 and explain why it works.

Problem 5: Show that a tree has at most one perfect matching.
Hint: Given two perfect matchings, think about the symmetric difference.

Problem 6: Let $k \in \mathbb{N}^+$. A k-critical graph is a graph G such that $\chi(G) = k$ but $\chi(H) < k$ for any proper subgraph H of G, i.e. G has chromatic number k, but whenever we delete at least one vertex/edge, the chromatic number becomes strictly smaller. For example, every odd cycle is 3-critical (and in fact these are the only 3-critical simple graphs).
a. Show that K_n is an n-critical graph for each $n \in \mathbb{N}^+$.
b. Suppose that G is a k-critical graph. Show that $d(v) \geq k - 1$ for all $v \in V$.
Hint for b: Suppose that $v \in V$ is such that $d(v) \leq k - 2$. Think about the graph $G - v$.

***Problem 7:** (Challenge Problem - won't be graded.) One can ask if Hall's Theorem (Theorem 11.13) works for infinite bipartite graphs. In doing so, one should interpret $|S| \leq |N(S)|$ as saying that if S is infinite, then $N(S)$ is also infinite. Show that the result is false by constructing an bipartite graph G on with vertex set $\mathbb{N}$ such that $|S| \leq |N(S)|$ for all $S \subseteq X$, but for which there is no matching that saturates X.