Homework 12 : Due Wednesday, November 16

Problem 1: Let T be the unique tree with vertex set $[8]$ whose Prüfer code is $5, 2, 2, 5, 1, 2$. Find the corresponding sequence $a_1, a_2, \ldots, a_7$ and then draw T.

Problem 2: A saturated hydrocarbon is a molecule $C_k H_\ell$ in which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds forms a cycle. Show that $\ell = 2k + 2$ in any saturated hydrocarbon.

Problem 3: Let T be a tree, and suppose that T has k vertices of degree at least 3. Show that T has at least $k + 2$ many leaves.

Problem 4:
a. Count the number of trees with vertex set $[n]$ having exactly 2 leaves.
b. Count the number of trees with vertex set $[n]$ having exactly $n - 2$ leaves.
c. Count the number of trees with vertex set $[11]$ where all of the following hold:

- $d(5) = 4$
- $d(1) = d(7) = 3$
- $d(4) = d(8) = 2$
- $d(v) = 1$ for all other vertices, i.e. all other vertices are leaves.

Hint: You can count some of these directly, but it is easier to think about Prüfer codes.

Problem 5: Let T be a tree and let v be a vertex of T. Let $T - v$ be the subgraph of T obtained by deleting v and all edges incident to it. Show that if v is not a leaf of T, then $T - v$ is not connected.

Problem 6: Let G be a connected graph with at least 2 vertices. Show that there exist distinct vertices u and w such that both $G - u$ and $G - w$ are connected.

Problem 7:
a. Let G be a graph with n vertices and at least n edges. Show that G has a cycle.
b. For each n with $1 \leq n \leq 4$, construct a simple graph G with n vertices such that both G and $\overline{G}$ are acyclic.
c. Show that if G is a simple graph on $n \geq 5$ vertices, then at least one of G or $\overline{G}$ has a cycle.