Homework 11 : Due Friday, November 11

Problem 1: Suppose that G is a graph with exactly two vertices u and w of odd degree. Show that there is a u, w-path in G.

Problem 2: Let G be a simple graph. Define a new simple graph $\overline{G}$, called the complement of G, as follows. Let $V_{\overline{G}} = V_G$, i.e. the vertex set of $\overline{G}$ is the vertex set of G. Given two distinct vertices u and w, include an edge in $\overline{G}$ with endpoints u and w exactly when no such edge exists in G.

Show that if G is a simple disconnected graph, then $\overline{G}$ is connected.

Problem 3: Given a path in a graph G, we define its *length* to be the number of edges it contains. In other words, if

$$v_0\ e_1\ v_1\ e_2\ v_2\ \dots e_n v_n$$

is a path, we define its length to be n. Let G be a connected graph, and let P and Q be two paths of maximum length in G (so both P and Q have common length n, and there is no path of length greater than n). Show that P and Q have a common vertex.

Problem 4: Let $n \in \mathbb{N}^+$. Determine the maximum number of edges that a simple disconnected graph on n vertices can have.
Hint: Depending on your approach, some calculus might be useful.

Problem 5: Let $n \in \mathbb{N}$ with $n \geq 2$. Notice that the number of simple graphs with vertex set $[n]$ is

$$2^{\binom{n}{2}}$$

(where the $\binom{n}{2}$ is in the exponent) because for each of the $\binom{n}{2}$ many pairs of vertices, we need to decide whether there is an edge with those endpoints. Show that the number of simple graphs with vertex set $[n]$ where every vertex has even degree is

$$2^{\binom{n-1}{2}}$$

Hint: Establish a bijection with a set that you know how to count.

Problem 6: Use induction (on the number of vertices) to show that if T is a tree having a vertex of degree Δ, then T has at least Δ leaves.