Homework 10 : Due Monday, November 7

Problem 1: Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.
a. $A = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.
b. $A = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.
c. $A = \mathbb{N} \times \mathbb{N}$ where $(a, b) \sim (c, d)$ means $a + d = b + c$.

Problem 2: Fix $n \in \mathbb{N}^+$. Define a relation $\sim$ on $\mathbb{Z}$ by saying that $a \sim b$ if $n \mid (a - b)$.
a. Show that $\sim$ is an equivalence relation.
b. Show that $a \sim b$ if and only if a and b leave the same unique remainder upon division by n. Note that you must prove both directions here because the statement is "if and only if".
c. How many distinct equivalence classes does $\sim$ have? Explain.

Problem 3: A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

> "If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property."

Now you know that their argument must be wrong because one of the examples in Problem 1 is symmetric and transitive but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.

Problem 4: Fix $n \in \mathbb{N}^+$. Consider the simple graph Q_n defined as follows. Let the vertex set V be the set of all ordered n-tuples of 0's and 1's (so for example, if $n = 3$, then one vertex is $(0, 1, 0)$ and another is $(1, 1, 1)$). For the edge set E, given $u, v \in V$, include an edge with endpoints u and v when they differ in exactly one coordinate (so for example when $n = 3$ there is edge with endpoints $(0, 0, 1)$ and $(1, 0, 1)$). The simple graph Q_n is called is called the n-cube. Determine the number of vertices and edges in Q_n.

Problem 5: Determine the maximum number of edges that a simple graph on n vertices can have. Explain.

Problem 6: Let G be a graph with n vertices and k edges. Let δ be the minimum degree of any vertex in G, and let Δ be the maximum degree of any vertex in G. Show that

$$\delta \leq \frac{2k}{n} \leq \Delta$$

Problem 7: Let G be a graph. Let G' be the graph obtained by taking two vertices u and v in V_G and adding a new edge e with endpoints u and v. Thus, G' has the same vertex set as G but includes one new edge. Show that if G has k connected components and G' has ℓ connected components, then either $\ell = k$ or $\ell = k - 1$.

Problem 8: Show that if G is a connected graph with n vertices, then G must have at least $n - 1$ edges.