Writing Assignment 5: Due Wednesday, March 6

Problem 1: Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and that $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. Show that if $\mathsf{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, then $\mathsf{Span}(T(\vec{u}_1), T(\vec{u}_2)) = \mathbb{R}^2$.
Hint: You are trying to prove that two sets are equal, so you should naturally think about a double containment proof. However, you really need only show that $\mathbb{R}^2 \subseteq \mathsf{Span}(T(\vec{u}_1), T(\vec{u}_2))$, because the other containment is immediate.

Problem 2: Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an injective linear transformation and that $\vec{u}, \vec{w} \in \mathbb{R}^2$. Show that if $\vec{w} \notin \mathsf{Span}(\vec{u})$, then $T(\vec{w}) \notin \mathsf{Span}(T(\vec{u}))$.

Problem 3:
a. Suppose that $\vec{u}, \vec{w} \in \mathbb{R}^2$ are both nonzero vectors. Show that if $\vec{w} \in \mathsf{Span}(\vec{u})$, then $\mathsf{Span}(\vec{w}) = \mathsf{Span}(\vec{u})$.
b. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose that $ad - bc = 0$ and at least one of a, b, c, d is nonzero. Show that if $\vec{w} \in \mathsf{Null}(T)$ and $\vec{w} \neq \vec{0}$, then $\mathsf{Null}(T) = \mathsf{Span}(\vec{w})$.
Hint for (a): Make use of Problem 5 on Problem Set 6 to avoid doing everything from scratch.
Note for (b): We know from Theorem 2.7.3 that there exists $\vec{u} \in \mathbb{R}^2$ with $\mathsf{Null}(T) = \mathsf{Span}(\vec{u})$. But this problem is asking you to show something stronger: if you take *any* nonzero vector in $\mathsf{Null}(T)$, then $\mathsf{Null}(T)$ is the span of that vector.