Writing Assignment 4: Due Wednesday, February 28

Problem 1: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$\mathsf{range}(T) = \{\vec{w} \in \mathbb{R}^2 : \text{There exists } \vec{v} \in \mathbb{R}^2 \text{ with } \vec{w} = T(\vec{v})\}.$$

Notice that $\vec{0} \in \mathsf{range}(T)$ because we know that $T(\vec{0}) = \vec{0}$ by Proposition 2.4.2.
a. Show that if $\vec{w}_1, \vec{w}_2 \in \mathsf{range}(T)$, then $\vec{w}_1 + \vec{w}_2 \in \mathsf{range}(T)$.
b. Show that if $\vec{w} \in \mathsf{range}(T)$ and $c \in \mathbb{R}$, then $c\vec{w} \in \mathsf{range}(T)$.

Problem 2: Let $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{R}^2$. Show that at least one of the $\vec{u}_i$ is in the span of the other two. That is show that either $\vec{u}_1 \in \mathsf{Span}(\vec{u}_2, \vec{u}_3)$, or $\vec{u}_2 \in \mathsf{Span}(\vec{u}_1, \vec{u}_3)$, or $\vec{u}_3 \in \mathsf{Span}(\vec{u}_1, \vec{u}_2)$.
Hint: Do some cases. What can you conclude if $(\vec{u}_1, \vec{u}_2)$ is a basis of $\mathbb{R}^2$? If not, what does Theorem 2.3.10 tell you about $\vec{u}_1$ and $\vec{u}_2$?

Problem 3: We defined linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, but we can also define them from $\mathbb{R}$ to $\mathbb{R}$ as follows. A linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is a function $f\colon \mathbb{R} \to \mathbb{R}$ with both of the following properties:

- $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
- $f(c \cdot x) = c \cdot f(x)$ for all $c, x \in \mathbb{R}$.

Given any $r \in \mathbb{R}$, it is straightforward to check that the function $g_r\colon \mathbb{R} \to \mathbb{R}$ given by $g_r(x) = rx$ is a linear transformation (no need to do this). Show that these are the only linear transformations from $\mathbb{R}$ to $\mathbb{R}$. In other words, show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a linear transformation, then there exists $r \in \mathbb{R}$ with $f = g_r$.
Hint: Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a linear transformation. If you know $f(1)$, can you determine $f(4)$? How about $f(\pi)$?