Problem Set 7: Due Friday, February 16

Problem 1:
a. Show that

$$\left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$$

is a basis of $\mathbb{R}^2$.
b. Explicitly compute the unique values of $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 12 \\ 7 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

c. Explicitly compute the unique values of $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 13 \\ 25 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Problem 2: Find, with explanation, all values of $c \in \mathbb{R}$ such that

$$\left(\begin{pmatrix} c-3 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 \\ c \end{pmatrix}\right)$$

is a basis of $\mathbb{R}^2$.

Problem 3: Let

$$A = \left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R}\right\} \quad \text{and} \quad B = \left\{\begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R}\right\}.$$

In this problem, we will work through the outline of how to show that $A = B$ via a double containment proof.

a. Let's show that $A \subseteq B$. Let $\vec{u} \in A$ be arbitrary. By definition of A, we can fix $c \in \mathbb{R}$ with

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

To show that $\vec{u} \in B$, we have to fill in the blank of

$$\vec{u} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + \underline{\quad} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

with a real number. What real number (that depends on c) works? Justify your choice.

b. Let's show that $B \subseteq A$. Let $\vec{u} \in B$ be arbitrary. By definition of B, we can fix $c \in \mathbb{R}$ with

$$\vec{u} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

To show that $\vec{u} \in A$, we have to fill in the blank of

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \underline{\quad} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

with a real number. What real number (that depends on c) works? Justify your choice.

Problem 4: In each of the following cases, determine whether the given function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. If yes, explain why. If no, provide an explicit counterexample.

a. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+7y \\ 5x-4y \end{pmatrix}$.

b. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} xy \\ x+y \end{pmatrix}$.

c. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y\sin^2(x^3) + y\cos^2(x^3)) \\ y \end{pmatrix}$.

Problem 5: Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ -5x+3y \end{pmatrix}.$$

Show that

$$\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in \mathsf{range}(T)$$

by explicitly finding $\vec{v} \in \mathbb{R}^2$ with

$$T(\vec{v}) = \begin{pmatrix} -18 \\ 47 \end{pmatrix}.$$