Problem Set 5: Due Friday, February 9

Problem 1: Each of the following is an *attempted* description of a set, but some of them do not make sense. In each part, explain whether the definition is valid or not. When it is a valid description, give 3 specific elements of the corresponding set.
a. $\{n^2 + 19m : n, m \in \mathbb{N}\}$.
b. $\{x \in \mathbb{Q} : x^2 + 5x - 3\}$.
c. $\{n \in \mathbb{Z} : 5 < 2^n \text{ and } 10n < 81\}$.
d. $\{x^2 + 6 \in \mathbb{R} : x - 1 > 0\}$.

Problem 2: Define $f \colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = e^x$. Write down a description of the set $\mathsf{range}(f)$ by carving it out of a set using a property without any quantifiers. Briefly explain why your set is equal (no need to give a formal proof).

Problem 3: Consider the function $f \colon \mathbb{Q} \to \mathbb{Q}$ given by $f(a) = 3a + 2$. We clearly have $\mathsf{range}(f) \subseteq \mathbb{Q}$ by definition. Thus, to show that $\mathbb{Q} = \mathsf{range}(f)$, it suffices to show that $\mathbb{Q} \subseteq \mathsf{range}(f)$. To do this, we need to show how to take an arbitrary $b \in \mathbb{Q}$, and fill in the blank in $f(\underline{\quad\quad}) = b$ with an element of $\mathbb{Q}$. In this problem, we first do a few examples, and then handle a general b.
a. Fill in the blank in $f(\underline{\quad\quad}) = 2$ with an element of $\mathbb{Q}$.
b. Fill in the blank in $f(\underline{\quad\quad}) = -19$ with an element of $\mathbb{Q}$.
c. Fill in the blank in $f(\underline{\quad\quad}) = 55$ with an element of $\mathbb{Q}$.
d. Let $b \in \mathbb{Q}$ be arbitrary. Fill in the blank in $f(\underline{\quad\quad}) = b$ with an element of $\mathbb{Q}$ (your answer will depend on b), and justify that your choice works.

Problem 4: Define $f \colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $f(n)$ be the number of positive divisors of n. Define $g \colon \mathbb{N}^+ \to \mathbb{N}$ by letting $g(n)$ be the number of primes less than or equal to n. For example, we have $g(1) = 0$, $g(2) = 1$, and $g(6) = 3$.
a. Calculate, with explanation, the values of $(g \circ f)(6)$ and $(g \circ f)(36)$.
b. Find an example, with explanation, of an $n \in \mathbb{N}^+$ with $(g \circ f)(n) = 3$.

Problem 5: Let $A = \{1, 2\}$. Give an example, with explanation, of two functions $f \colon A \to A$ and $g \colon A \to A$ such that $f \circ g \neq g \circ f$.

Problem 6: Define a function $f \colon \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ as follows. On input $n \in \{0, 1, 2, 3, 4\}$, let $f(n)$ be the remainder that arises when you divide the number $3n$ by 5. Is f injective, surjective, both, or neither? Explain.

Problem 7: Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 8x$. Show that f is not injective.