Problem Set 3: Due Friday, February 2

Note: In Problems 1, 2, and 3, write up the whole proof, not just the pieces that go into the blanks. Also, fill in as much as necessary in the blanks so that no steps are omitted. Finally, if you would prefer to write the argument differently in your words, please feel free to do so.

Problem 1: In this problem, we write a careful proof of a fact that we discussed on the second day of class. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $x, y \in \mathbb{Z}$, we have that $22x + 14y$ is even":

Let $x, y \in \mathbb{Z}$ be arbitrary. We have $22x + 14y =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $22x + 14y$ is even.

Problem 2: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "For all $a \in \mathbb{Z}$, we have that $2a^3 + 6a - 3$ is odd":

Let $a \in \mathbb{Z}$ be arbitrary. We have $2a^3 + 6a - 3 =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $2a^3 + 6a - 3$ is odd.

Problem 3: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a + b$ is even".

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can fix ________________. Since b is odd, we can fix ________________. Now notice that $a + b =$ ________________. Since ________________ $\in \mathbb{Z}$, we conclude that $a + b$ is even.

Problem 4: Consider the statement "If $a \in \mathbb{Z}$ is odd, then $4a + 1$ is even". Your friend claims to have a proof, and presents the following argument:

Let $a \in \mathbb{Z}$ be an arbitrary odd integer. Since a is odd, we can fix $m \in \mathbb{Z}$ with $a = 2m + 1$. We then have that

$$\begin{aligned} 4a + 1 &= 4 \cdot (2m + 1) + 1 \\ &= 8m + 5 \\ &= 2 \cdot \left(\frac{8m + 5}{2} \right). \end{aligned}$$

We have shown the existence of an $n \in \mathbb{Z}$ with $4a + 1 = 2n$, so $4a + 1$ is even.

a. Pinpoint the error in your friend's argument. Be as specific as you can.
b. Is the statement in question true or false? Justify your answer carefully.

Problem 5: Consider the following statement: For all $a \in \mathbb{R}$, if $a < 1$, then $a^2 < 1$.
a. Write the negation of each of the the given statement so that no "not" appears.
b. Show that the given statement is false by arguing that its negation is true.
Note: Be very careful on part (a). Don't just mindlessly apply symbolical manipulations! Think through the logic carefully. What precisely would count as a counterexample to an "if...then..." statement? It might help to rewrite the "if...then..." statement using and/or/not, to look at Problem 4, or to consider the truth table at the top of p. 24 of the book.

Problem 6: Write both the converse and contrapositive of each of the following statements (no need to argue whether any of the them are true or false). In each case, get rid of all occurrences of *not* in the final result.

a. If $a \in \mathbb{Z}$ and $a \geq 2$, then $4a > 7$.

b. If $x, y \in \mathbb{R}$ and $x^4 + y^4 = 1$, then $x^2 + y^2 \leq 2$.

c. If $a \in \mathbb{Z}$ and there exists $m \in \mathbb{Z}$ with $a = 10m$, then there exists $m \in \mathbb{Z}$ with $a = 5m$.