Problem Set 20: Due Friday, April 26

Problem 1: Working in $\mathbb{R}^4$, find the dimension of the subspace

$$W = \mathsf{Span}\left(\begin{pmatrix}1\\5\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\-6\\1\\3\end{pmatrix}, \begin{pmatrix}2\\8\\2\\2\end{pmatrix}\right).$$

Problem 2: Working in the vector space $\mathcal{F}$ of all functions $f\colon \mathbb{R} \to \mathbb{R}$, define the following:

- $f_1(x) = 2^x$.
- $f_2(x) = 3^x$.

Find, with explanation, the dimension of the subspace $W = \mathsf{Span}(f_1, f_2)$.

Problem 3: Define $T\colon \mathcal{P}_1 \to \mathbb{R}^2$ by letting

$$T(a+bx) = \begin{pmatrix}a-b\\b\end{pmatrix}.$$

Show that T is a linear transformation.

Problem 4: Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x-y\\x+z\\y+z\end{pmatrix}.$$

a. Explain why T is a linear transformation.
b. Give an example of a nonzero $\vec{v} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{0}$.
c. Show that T is not injective.

Problem 5: Define $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ by letting

$$T(f) = \begin{pmatrix}f(0)\\f(2)\end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for $\mathcal{P}_2$.
a. Let

$$\varepsilon_2 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^{\varepsilon_2}$?
b. Let

$$\beta = \left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right),$$

which is a basis of $\mathbb{R}^2$. What is $[T]_\alpha^\beta$?