Problem Set 2: Due Monday, January 29

Problem 1: Let $A = \{1, 2, 3, 4\}$. Suppose that $P(x, y)$ and $Q(x, y)$ are expressed by the following tables:

P	1	2	3	4
1	F	T	T	F
2	F	T	F	T
3	T	F	F	F
4	F	F	F	F

Q	1	2	3	4
1	F	T	T	T
2	F	T	T	T
3	T	F	T	F
4	F	F	T	T

Interpret the diagrams as follows. Given $x, y \in \{1, 2, 3, 4\}$, to determine the truth value of $P(x, y)$, go to row x and column y. For example, $P(1, 2)$ is true and $P(3, 3)$ is false. Determine, with careful explanation, whether each of the following are true or false.
a. There exist $x, y \in A$ with $\text{Not}(x = y)$, $P(x, y)$, and $P(y, x)$.
b. For all $x, y \in A$, we have $\text{Not}(Q(x, y))$.
c. For all $x, y \in A$, if $P(x, y)$, then $Q(x, y)$.
d. For all $x, y \in A$, either $Q(x, y)$ or $Q(y, x)$.
e. For all $x \in A$, there exists $y \in A$ with $\text{Not}(P(x, y))$.
f. There exists $x \in A$ such that for all $y \in A$, we have $P(x, y)$.
g. There exists $y \in A$ such that for all $x \in A$, we have $Q(x, y)$.

Problem 2: Let A be the set of students in the course. Given $x \in A$, let $C(x)$ be true if x has a cat, let $D(x)$ be true if x has a dog, and let $R(x)$ be true if x has a rabbit. Express each of the following statements carefully using only A, C, D, R, our quantifiers "for all" and "there exists", and the connectives "=" "and", "or", "not", and "if...then...". In other words, express each of the following like the statements in Problem 1.
a. At least two students in the class have a dog.
b. Nobody in the class has all three types of pet.
c. Every student who has a rabbit also has (at least) one of the other two types of pet.

Problem 3: Given $x, y \in \mathbb{N}^+$, let $D(x, y)$ be true if x divides evenly into y. Express each of the following statements carefully using only $\mathbb{N}^+$ and D, our quantifiers "for all" and "there exists", and the connectives "=" "and", "or", "not", and "if...then...". In other words, express each of the following like the statements in Problem 1. You do *not* need to explain if the statements are true or false.
a. The number 1 divides evenly into all numbers.
b. Whenever x divides evenly into y, and y divides evenly into z, it follows that x divides evenly into z.
c. No number has the property that every number divides evenly into it.
d. The numbers 1 and 91 both divide evenly into 91, and no other number divides evenly into 91.

Problem 4: Let A be the set of all people. Given $x, y \in A$, let $M(x, y)$ be true if x has ever sent a text message to y, and be false otherwise. Express each of the following statements carefully using only A and M, our quantifiers "for all" and "there exists", and the connectives "=" "and", "or", "not", and "if...then...". In other words, express each of the following like the statements in Problem 1.
a. Everybody has sent a text message to somebody.
b. Somebody has never received a text message from anyone.
c. Somebody has sent a text message to (at least) two different people.
d. Somebody has sent a text message to each person that they have received a message from.

Problem 5: Write the negation of each of the the following statements so that no "not" appears. You do *not* need to explain if the statements are true or false.

a. For all $x \in \mathbb{R}$, we have $e^x \neq 0$.
b. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, we have $x + y^2 \geq 3$.
c. For all $x, y \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that both $x < q$ and $q < y$.
d. There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$.