Problem Set 15: Due Friday, April 5

Problem 1: Let $V = \mathbb{R}^3$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix}.$$

Show that there is no element of V that serves as $\vec{0}$. That is, show that the statement

"There exists $\vec{z} \in V$ such that for all $\vec{v} \in V$, we have $\vec{v} + \vec{z} = \vec{v}$"

is false.

Problem 2: Let $V = \mathbb{R}^2$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ a_2 \end{pmatrix}.$$

Also, let

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Show that V is not a vector space by explicitly finding a counterexample to one of the 10 properties.

Problem 3: Let V be a vector space. Show that $\vec{u} + (\vec{v} + \vec{w}) = \vec{w} + (\vec{v} + \vec{u})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$. Carefully state what property you are using in every step of your argument.

Problem 4: Let

$$W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}.$$

Show that W is a subspace of $\mathbb{R}^3$.

Problem 5: Recall that $\mathcal{P}$ is the vector space of all polynomial functions $f : \mathbb{R} \to \mathbb{R}$. Let W be the subset of $\mathcal{P}$ consisting of those polynomials that have a nonnegative constant term (i.e. the constant terms is greater than or equal to 0). Is W a subspace of $\mathcal{P}$? Either prove or give a counterexample.

Problem 6: Let V be the vector space of all 2×2 matrices. Show that

$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \mathsf{Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix} \right).$$