Problem Set 14: Due Friday, March 15

Problem 1: Given a nonzero vector $\vec{w} \in \mathbb{R}^2$, we discussed the linear transformation $F_{\vec{w}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ that reflects the plane across the line $\mathsf{Span}(\vec{w})$ in Proposition 2.5.11. Compute $\det([F_{\vec{w}}])$. Simplify your answer as much as possible.

Problem 2: Show that $\det(AB) = \det(A) \cdot \det(B)$ for all 2×2 matrices A and B.
Note: Intuitively, if the linear transformation with standard matrix B distorts area by a factor of s, and the linear transformation with standard matrix A distorts area by a factor of r, then the *composition* of these linear transformations will distort area by a factor of rs (because matrix multiplication corresponds to function composition), with appropriate signs. Although it is possible to make this geometric sketch precise by using arguments similar to the ones at the end of Section 3.4, you should give a computational argument in this problem by just using the formula for the determinant.

Problem 3: Show that if A is an invertible 2×2 matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Hint: Start with the fact that $AA^{-1} = I$, and use the previous problem.

Problem 4: Show that if $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and α is a basis of $\mathbb{R}^2$, then $\det([T]_\alpha) = \det([T])$. Thus, although we might obtain different matrices when we represent T with respect to different bases, the resulting matrices will all have the same determinant.

Problem 5: Given a 2×2 matrix A and an $r \in \mathbb{R}$, what is the relationship between $\det(r \cdot A)$ and $\det(A)$? Explain.