Problem Set 11: Due Monday, March 4

Problem 1: In this problem, let 0 denote the 2×2 zero matrix, i.e. the 2×2 matrix where all four entries are 0.
a. Give an example of a nonzero 2×2 matrix A with $A \cdot A = 0$.
b. Show that there does not exist an invertible 2×2 matrix A with $A \cdot A = 0$.
Hint: Suppose that you are working in the world of real numbers. How would you convince yourself that if $x^2 = 0$, then x would have to be 0? If $x \neq 0$, what can you do to both sides?

Problem 2: Find a 2×2 matrix A with

$$A \cdot A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hint: Working computationally is not advised. Think about a linear transformation that you can apply to the plane so that when you do it twice, you end up sending $(1, 0)$ to $(0, 1)$ and sending $(0, 1)$ to $(-1, 0)$.

Problem 3: Let A, B, C all be invertible 2×2 matrices. Must there exist a 2×2 matrix X with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.
Hint: Again, consider working in the world of numbers. If you have nonzero real numbers a, b, and c, must there exist a real number x with $a(x + b)c = 1$? If so, what is it? And how would you justify that it works?

Problem 4: Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}.$$

Explain why T has an inverse and calculate

$$T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right).$$

Problem 5: Consider the following system of equations:

$$\begin{array}{rcrcr} x & + & 4y & = & -3 \\ 2x & + & 5y & = & 8. \end{array}$$

a. Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix A and vector $\vec{b}$.
b. Explain why A is invertible and calculate A^{-1}.
c. Use A^{-1} to solve the system.

Problem 6: Let

$$\alpha = \left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}\right).$$

a. Show that α is a basis for $\mathbb{R}^2$.
b. Compute $\left[\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right]_\alpha$.
c. Compute $\left[\begin{pmatrix} 8 \\ 17 \end{pmatrix}\right]_\alpha$.
In each part, briefly explain how you carried out your computation.