Problem Set 10: Due Friday, March 1

Problem 1: Let $\theta \in \mathbb{R}$. Define $C_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $C_\theta(\vec{v})$ be the result of rotating $\vec{v}$ *clockwise* around the origin by an angle of θ. Explain why C_θ is a linear transformation and why

$$[C_\theta] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Hint: Can you interpret C_θ as a certain counterclockwise rotation?

Problem 2: Let $\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$ be a nonzero vector. Recall that we defined the function $P_{\vec{w}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ that projects points onto the line $\mathsf{Span}(\vec{w})$. By Proposition 2.5.10, we know that $P_{\vec{w}}$ is a linear transformation, and that it has standard matrix

$$A = \begin{pmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{pmatrix}.$$

a. Show that $A \cdot A = A$ by simply computing it.
b. By interpreting the action of $P_{\vec{w}}$ geometrically, explain why you should expect that $A \cdot A = A$.
Cultural Aside: A matrix A that satisfies $A \cdot A = A$ is called *idempotent*.

Problem 3: Define $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first reflecting $\vec{v}$ across the x-axis, and then reflecting the result across the y-axis.
a. Compute $[T]$.
b. The action of T is the same as a certain rotation. Explain which rotation it is.

Problem 4: Let A and B be 2×2 matrices. Assume that $A\vec{v} = B\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. Show that $A = B$.

Problem 5: Consider the unique linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}.$$

Find, with explanation, vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $\mathsf{Null}(T) = \mathsf{Span}(\vec{u})$ and $\mathsf{range}(T) = \mathsf{Span}(\vec{w})$.

Problem 6 : Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$\mathsf{Null}(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}.$$

a. Show that if $\vec{v}_1, \vec{v}_2 \in \mathsf{Null}(T)$, then $\vec{v}_1 + \vec{v}_2 \in \mathsf{Null}(T)$.
b. Show that if $\vec{v} \in \mathsf{Null}(T)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \mathsf{Null}(T)$.