Writing Assignment 5: Due Wednesday, March 5

Problem 1: Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and that $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. Show that if $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, then $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) = \mathbb{R}^2$.
Hint: You are trying to prove that two sets are equal, so you should naturally think about a double containment proof. However, you really need only show that $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$, because the other containment is immediate.

Problem 2: Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an injective linear transformation and that $\vec{u}, \vec{w} \in \mathbb{R}^2$. Show that if $\vec{w} \notin \text{Span}(\vec{u})$, then $T(\vec{w}) \notin \text{Span}(T(\vec{u}))$.

Problem 3: In this problem, we determine which 2×2 matrices commute with *every* 2×2 matrix.
a. Show that if $r \in \mathbb{R}$ and we let

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

then $AB = BA$ for every 2×2 matrix B.
b. Suppose that A is a 2×2 matrix with the property that $AB = BA$ for every 2×2 matrix B. Show that there exists $r \in \mathbb{R}$ such that

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Hint: For part (b), make strategic choices for B to make your life as simple as possible. I suggest thinking about matrices with lots of zeros.