Problem Set 9: Due Friday, February 28

Problem 1: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of reflecting the plane across the line $2x+y=0$.
a. Calculate $[T]$.
b. Calculate $T\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$.

Problem 2: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first projecting $\vec{v}$ onto the line $y = 3x$, and then projecting the result onto the line $y = 4x$. Explain why T is a linear transformation, and then calculate $[T]$.

Problem 3: Let $\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$ be a nonzero vector. Recall that we defined the function $P_{\vec{w}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ that projects points onto the line $\text{Span}(\vec{w})$. By Proposition 2.5.10, we know that $P_{\vec{w}}$ is a linear transformation, and that it has standard matrix

$$A = \begin{pmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{pmatrix}.$$

a. Show that $A \cdot A = A$ by simply computing it.
b. By interpreting the action of $P_{\vec{w}}$ geometrically, explain why you should expect that $A \cdot A = A$.
Cultural Aside: A matrix A that satisfies $A \cdot A = A$ is called *idempotent*.

Problem 4: Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first reflecting $\vec{v}$ across the x-axis, and then reflecting the result across the y-axis.
a. Compute $[T]$.
b. The action of T is the same as a certain rotation. Explain which rotation it is.

Problem 5: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. We know from Proposition 2.4.7 that $r \cdot T$ is a linear transformation. Show that if

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

In other words, if we define the multiplication of a matrix by a scalar as in Definition 2.6.3, then the standard matrix of $r \cdot T$ is obtained by multiplying every element of $[T]$ by r.

Problem 6: Let A be a 2×2 matrix. Verify each of the following using the formula for the matrix-vector product.
a. $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.
b. $A(c \cdot \vec{v}) = c \cdot A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$ and all $c \in \mathbb{R}$.
Note: Since matrices encode linear transformations, you should expect these to be true. In fact, we can argue that they are true by interpreting the matrix as being the standard matrix of a certain linear transformation, and then just appealing to the fact that linear transformation preserve addition and scalar multiplication. However, in this problem, you should just work through the computations directly.